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LETTER TO THE EDITOR

The η invariant for charged spinors in Taub–NUT

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Abstract. The Atiyah–Singer index theorem for the Dirac operator on a manifold with boundary involves a non-local term constructed from the eigenvalues of the Dirac operator on the boundary. This η invariant is evaluated for charged spinors on the left-invariant S^3 boundary of the Taub–NUT instanton. It is shown that the index theorem is then in agreement with a previous explicit evaluation of the index in Taub–NUT.

In a compact four-dimensional Riemannian manifold M without boundary, the Atiyah–Singer index theorem for the Dirac operator for charged spinors is

$$n_+ - n_- = \frac{1}{192\pi^2}\int_M \mathrm{Tr}\,Ⓗ \wedge Ⓗ + \frac{e^2}{8\pi^2}\int_M F\wedge F \tag{1}$$

where $n_\pm$ are the numbers of L^2 solutions of the charged Dirac equation of positive (negative) chirality, Ⓗ is the matrix valued curvature two-form of the manifold, F is the Maxwell two-form and e is the charge of the spinor fields (see, for example, Eguchi *et al* 1980 and references therein). If the manifold has a boundary ∂M then there are extra boundary correction terms to be added to the right-hand side of equation (1):

$$-\frac{1}{192\pi^2}\int_{\partial M} \mathrm{Tr}\,\theta\wedge Ⓗ - \frac{e^2}{8\pi^2}\int_{\partial M} A\wedge F - \tfrac{1}{2}(\eta(0)+h) \tag{2}$$

where θ is the second fundamental form of ∂M in M, A is the electromagnetic potential, and $\eta(0)$ and h are non-local terms depending only on the boundary ∂M, which are the terms of interest in this paper (Eguchi *et al* 1980).

The η *invariant* $\eta(0)$ is the analytic continuation to $s=0$ of the meromorphic function $\eta(s)$ defined for $R(s)>2$ by

$$\eta(s) = \sum_{\lambda\neq 0} |\lambda|^{-2}\,\mathrm{sign}\,\lambda \equiv \sum_{\lambda>0}\lambda^{-s} - \sum_{\lambda<0}(-\lambda)^{-s} \tag{3}$$

where the sum is taken over the non-zero eigenvalues λ of the charged Dirac operator on the boundary manifold ∂M. h is the number of zero eigenvalues of the operator. We note for future reference that $\eta(0)$ is left invariant by a constant rescaling of all the eigenvalues, and hence by a constant conformal rescaling of the metric on ∂M.

In an earlier paper, the zero modes of the charged Dirac operator were investigated in the Taub-NUT gravitational instanton, in the presence of a self-dual electromagnetic field (Hawking 1977, Pope 1978). The metric may be given as

$$ds^2 = \left(\frac{r+M}{r-M}\right) dr^2 + (r^2 - M^2)(\sigma_1^2 + \sigma_2^2) + 4M^2\left(\frac{r-M}{r+M}\right)\sigma_3^2 \tag{4}$$

where $r \geq M$ and the σ_i are a basis of left-invariant one-forms on the three-sphere, which may be parametrised by Euler angles (θ, ϕ, ψ) as

$$\begin{aligned} \sigma_1 &= \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi \\ \sigma_2 &= -\sin\psi\, d\theta + \cos\psi \sin\theta\, d\phi \\ \sigma_3 &= d\psi + \cos\theta\, d\phi \end{aligned} \tag{5}$$

where $0 \leq \phi \leq 2\pi$, $0 \leq \psi \leq 4\pi$. The apparent singularity at $r = M$ is just a removable metric singularity, and the manifold is non-compact and regular, with the topology R^4 (Hawking 1977). It may be compactified by cutting off the radial coordinate r at some large distance r_0, thereby introducing a boundary $\partial M = S^3$ whose induced metric is

$$ds^2 = 4(r_0^2 - M^2)[\tfrac{1}{4}(\sigma_1^2 + \sigma_2^2) + \tfrac{1}{4}4M^2(r_0 + M)^{-2}\sigma_3^2]. \tag{6}$$

The self-dual electromagnetic field is

$$F = k\left[\frac{2M}{(r+m)^2} dr \wedge \sigma_3 - \left(\frac{r-M}{r+M}\right)\sigma_1 \wedge \sigma_2\right] \tag{7}$$

which may be derived from the potential

$$A = k\left(\frac{r-M}{r+M}\right)\sigma_3. \tag{8}$$

Because the topology of the manifold is trivial, the integral of F over any closed two-surface is zero, and so there is no Dirac quantisation condition, which means that the constant k is arbitrary. Setting $k = \beta/2e$ for convenience, one finds that $e^2/8\pi^2 \int_M F \wedge F = \frac{1}{2}\beta^2$. The curvature contribution to (1) is $-\frac{1}{12}$, and the surface integrals in (2) vanish, so

$$n_+ - n_- = -\tfrac{1}{12} - \tfrac{1}{2}(\eta(0) + h). \tag{9}$$

In Pope (1978) it was shown by explicit calculation of the zero modes that

$$n_+ - n_- = \tfrac{1}{2}n(n+1) \tag{10}$$

where $n = [\beta]$, the greatest integer less than β (we are taking β to be positive, without loss of generality). In this paper we reconcile (10) with the index theorem result (9), by calculating $\eta(0)$ using the method of Hitchin (1974).

The approach will be to calculate the eigenvalues of the charged Dirac operator in the left-invariant metric

$$ds^2 = \tfrac{1}{4}(\sigma_1^2 + \sigma_2^2) + \tfrac{1}{4}\mu^2\sigma_3^2 \tag{11}$$

on the three-sphere, and use these to evaluate $\eta(0)$ in the limit $\mu \to 0$, which can be seen from equation (6) to give the same limit (up to a constant conformal rescaling, which does not alter $\eta(0)$) as in the Taub–NUT instanton when the boundary is sent to infinity ($r_0 \to \infty$).

To find the eigenvalues of the Dirac operator P on S^3, $P\Psi = \lambda\Psi$, we introduce the notion of a *spinor-valued* zero-form Ψ, which may be represented as a two-component column vector $\binom{u}{v}$. Exterior differentiation of Ψ is given by

$$D\Psi = \mathrm{d}\Psi + \sigma\Psi \tag{12}$$

where $\sigma = \frac{1}{4}\tau_a\tau_b\omega_{ab}$, the spin connection; τ_a are the Pauli matrices and ω_{ab} are the connection forms for (11). In the orthonormal triad basis

$$e^1 = \tfrac{1}{2}\sigma_1 \qquad e^2 = \tfrac{1}{2}\sigma_2 \qquad e^3 = \tfrac{1}{2}\mu\sigma_3 \tag{13}$$

$$\sigma = \frac{-\mathrm{i}}{2}\mu\begin{pmatrix} 0 & e^1 - \mathrm{i}e^2 \\ e^1 + \mathrm{i}e^2 & 0 \end{pmatrix} + \frac{\mathrm{i}(\mu^2-2)}{2\mu}\begin{pmatrix} e_3 & 0 \\ 0 & -e_3 \end{pmatrix}. \tag{14}$$

The uncharged Dirac operator P acting on Ψ is thus

$$P\Psi = \mathrm{i}\tau_a\langle D\psi, e^a\rangle = \mathrm{i}\tau_a e^a(\Psi) + \frac{\mu^2+2}{\mu}\Psi \tag{15}$$

where $e^a(\Psi)$ means ordinary differentiation of the scalar components u and v of Ψ. If Σ^a are the vectors dual to the left-invariant one-forms σ_a, then defining 'quantum mechanical' self-adjoint operators by $K^a = \mathrm{i}\Sigma^a$, and setting $K_\pm = K_1 \pm \mathrm{i}K_2$, the Dirac operator may be written as

$$P = 2\begin{pmatrix} \mu^{-1}K_3 & K_- \\ K_+ & -\mu^{-1}K_3 \end{pmatrix} + \frac{\mu^2+2}{2\mu}. \tag{16}$$

The charged Dirac operator P_A is obtained by making the replacement $D \to D - \mathrm{i}eA$ in (15), and since we are interested in the case $A = k\sigma_3 = (\beta/2e)\sigma_3$ (see equation (8)),

$$P_A = \begin{pmatrix} \mu^{-1}(2K_3 - \beta) & 2K_- \\ 2K_+ & -\mu^{-1}(2K_3 - \beta) \end{pmatrix} + \frac{\mu^2+2}{2\mu}. \tag{17}$$

$K_\pm$ and K_3 satisfy the usual commutation relations for angular momentum generators, and so adopting the notation $|s\rangle$ for a ${}_sY_{lm}$ spin-spherical harmonic (Goldberg *et al* 1967), the eigenvectors of P_A may be written as

$$\begin{pmatrix} a|s\rangle \\ b|s+1\rangle \end{pmatrix} \tag{18}$$

for some constants a and b. The actions of $K_\pm$, K_3 on $|s\rangle$ are

$$K_\pm|s\rangle = [(l \mp s)(l \pm s + 1)]^{1/2}|s+1\rangle \qquad K_3|s\rangle = s|s\rangle \tag{19}$$

with $|s| \leq l$, $|m| \leq l$. l may take integer or half-integer values. Thus for $-l \leq s \leq l-1$, one finds the eigenvalues λ or P_A are

$$\lambda = \tfrac{1}{2}\mu \pm \mu^{-1}[(2s+1-\beta)^2 + 4\mu^2(l-s)(l+s+1)]^{1/2} \tag{20}$$

with degeneracy $d = 2l+1$ for each permitted value of s.

There are also two 'exceptional' cases, when $s = l$ or $s = -(l+1)$, for which respectively b or a in (18) vanish, with eigenvalues

$$\lambda = \mu^{-1}(2l+1-\beta) + \tfrac{1}{2}\mu \qquad d = 2l+1 \tag{21}$$

$$\lambda = \mu^{-1}(2l+1+\beta) + \tfrac{1}{2}\mu \qquad d = 2l+1 \tag{22}$$

respectively.

We are interested in the $\mu \to 0$ limit. Since $\eta(0)$ is invariant under a constant simultaneous rescaling of all the eigenvalues, we may set $\lambda \to \mu\lambda$ before taking the limit

$\mu \to 0$, thereby obtaining finite eigenvalues. It is then clear that all the eigenvalues (20) will be symmetric between the positives and negatives in this limit making no net contribution to $\eta(s)$. The entire contribution to $\eta(s)$ comes from the exceptional cases (21) and (22), which after rescaling and setting $\mu = 0$, are

$$\lambda = p - \beta \qquad d = p \tag{23}$$

$$\lambda = p + \beta \qquad d = p \tag{24}$$

where $2l + 1 = p$, and so p takes integer values, $p > 1$. Without loss of generality we may take $\beta > 0$, and so for $\beta \neq$ integer if $n = [\beta]$, the integer part of β,

$$\eta(s) = \sum_{p=1}^{\infty} (p+\beta)^{-s} p + \sum_{p=n+1}^{\infty} (p-\beta)^{-s} p - \sum_{p=1}^{n} (\beta - p)^{-2} p. \tag{25}$$

Sums of the form of the first two terms in (25) may be evaluated at $s = 0$ by expanding $(p \pm \beta)^{-s}$ in descending powers of p, to obtain an infinite series of Riemann zeta functions, in which only a finite number of terms remain when s is set to zero. Thus

$$\eta(0) = -\tfrac{1}{6} + \beta^2 - n(n+1). \tag{26}$$

For the case that β is an integer, there will be β zero eigenvalues in equation (23), and so one finds that for all $\beta > 0$,

$$\eta(0) + h = -\tfrac{1}{6} + \beta^2 - n(n+1) \tag{27}$$

where $n = [\beta]$, the greatest integer less than β. Inserting this result into the index theorem (9), we recover the result (10) obtained by explicit calculation of Dirac zero modes in Taub–NUT.

Finally, we remark that this calculation may easily be extended to the case where S^3 is factored by the cyclic group Z_q to give the lens space $L(1, q)$. This means that the Euler angle coordinate ψ is now the identified modulo $4\pi/q$ (Gibbons *et al* 1979). The eigenfunctions on $L(1, q)$ are a subset of the S^3 eigenfunctions derived above; namely, those which are periodic with period $4\pi/q$. This periodicity must be analysed with respect to a non-rotating spinor dyad basis, and since the orthonormal triad rotates by $4\pi/q$ under $\psi \to \psi + 4\pi/q$, the components of Ψ with respect to a *non-rotating* dyad basis will be

$$\begin{pmatrix} a\mathrm{e}^{\mathrm{i}\psi/2} & |s\rangle \\ b\mathrm{e}^{-\mathrm{i}\psi/2} & |s+1\rangle \end{pmatrix} \tag{28}$$

rather than equation (18).

As before, only the exceptional eigenvalues (21) and (22) will contribute to $\eta(s)$, and from (28) these will have $2l + 1 = qp$, $p = \text{integer} \geq 1$. The calculation of $\eta(0) + h$ proceeds in a manner similar to the S^3 case, giving

$$\eta(0) = h = -\tfrac{1}{6}q + \beta^2/q - qn(n+1) \tag{29}$$

where $n = [\beta/q]$, the greatest integer less than β/q.

This result enables one to calculate the index for charged spinors in the multi-Taub–NUT background, with metric (Hawking 1977)

$$\mathrm{d}s^2 = V^{-1}(\mathrm{d}\tau + \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{x})^2 + V\,\mathrm{d}\boldsymbol{x}^2 \tag{30}$$

and self-dual Maxwell field generated by the potential

$$A = \beta V^{-1}(\mathrm{d}\tau + \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{x}) \tag{31}$$

where

$$V = 1 + \sum_{i=1}^{q} \frac{2M}{|x - x_i|} \qquad \nabla \wedge \omega = \nabla V. \tag{32}$$

Explicit calculation shows that the volume integral contributions (1) to the index are $-q/12 + \beta^2/2q$, which when combined with (29) give

$$n_+ - n_- = \tfrac{1}{2} qn(n+1) \tag{33}$$

which is of course an integer for all values of β.

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